CAVITATIONAL FLOW OF A PONDEROUS FLUID OVER A SMOOTH ARC

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This paper is concerned with the problem of constructing the cavitational flow of a ponderous fluid over a smooth arc for a given velocity distribution on the arc. Similar problems for an imponderable fluid were examined in [1-4]. The flow of a ponderous fluid over polygonal obstacles was investigated in [5,6] and in the linear formulation in [7,8].

§1. General solution of the problem. We consider the flow of a ponderous fluid, unbounded in the upward direction, over the smooth arc OA on a horizontal straight base in accordance with the Ryabushinskii model and with reflection (Fig. 1). We introduce the notation: V_∞ is the velocity of oncoming flow; V_0 is the velocity at the separation point A; S_0 is the arc length.

Let the velocity distribution on arc OA be given in the form of a function in the arc coordinate

$$V = V_0 f(s) \qquad (s = S/S_0,\ 0 \leqslant s \leqslant 1). \tag{1.1}$$

Here, $f(s)$ is a single-valued positive function satisfying a Hölder condition and the conditions $f(0) = 0$ and $f(1) = 1$.

It is necessary to construct the equation of the contour and the free boundary and to find the contour drag. Using symmetry, we consider only the second quadrant of the physical plane $z = x + iy$.

We introduce the Joukowski function

$$F = \ln\left(\frac{1}{V_0}\frac{dW}{dz}\right).$$

The solution of the problem can be reduced to finding the relations $W(\zeta)$ and $F(\zeta)$ where ζ varies in a certain canonical region. As this region we select the second quadrant of the plane $\zeta = \xi + i\eta$. We conformally map the flow region onto the region ζ with the point correspondence indicated in Figs. 1 and 2.

It is easy to see that the function which maps the region of variation of complex potential W onto the second quadrant of the ζ-plane has the form (Fig. 3):

$$W = \frac{\varphi_0 \zeta}{\sqrt{c^2 + \zeta^2}} \qquad (W = \varphi + i\psi). \tag{1.2}$$

Using (1.1) and (1.2) together with the relation $V = d\varphi/ds$, we obtain an equation for $s(\xi)$, which gives the correspondence between points on arc OA and points on ray $(-\infty, 1]$ of the ζ-plane

$$\frac{\varphi_0 \xi}{\sqrt{c^2 + \xi^2}} = -\varphi_0 + V_0 S_0 \int_0^s f(s)\, ds. \tag{1.3}$$

On the imaginary semiaxis of the ζ-plane the function $F(\zeta)$ is real and continuous. In accordance with the symmetry principle we continue it onto the entire upper half-plane. Now $F(\zeta)$ is definite and analytic in the upper half of the plane ζ and on the real axis it satisfies the conditions $\operatorname{Re}F(\xi) = \ln f(s(\xi))$, $|\xi| \geq 1$ and $\operatorname{Re}F(\xi) = u(\xi)$, $|\xi| \leq 1$.

Here, on the segment $[-1, 1]$ of the real axis ξ we have introduced the notation $F = u + iv$. The problem of finding a function $F(\zeta)$ in accordance with the above boundary conditions is solved by a Cauchy-type integral:

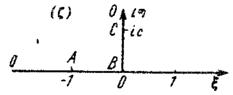

Fig. 2

$$F(\zeta) = \frac{1}{\pi i}\left[\int_{-\infty}^{-1} \frac{\ln f\, dt}{t - \zeta} + \int_1^{\infty} \frac{\ln f\, dt}{t - \zeta} + \int_{-1}^{1} \frac{u(t)}{t - \zeta}\, dt\right]. \tag{1.4}$$

If the function $u(\xi)$, $\xi \in [-1, 1]$ is known, (1.2) and (1.4) give the general solution. Thus, from (1.2), (1.4), and the expression for the Joukowski function, we obtain

$$z(\zeta) = \frac{\varphi_0 c^2}{V_0} \int_{-\infty}^{\zeta} \frac{e^{-F(\zeta)}}{(c^2 + \zeta^2)^{3/2}}\, d\zeta. \tag{1.5}$$

Hence, letting ζ tend to ξ, we obtain the equation of the contour,

$$z = \frac{\varphi_0 c^2}{V_0} \int_{-\infty}^{\xi} \frac{1}{f(s(\xi))(c^2 + \xi^2)^{3/2}}\, e^{i[q(\xi) + \Phi(u, \xi)]}\, d\xi,$$

$$q(\xi) = \frac{1}{\pi}\int_{-\infty}^{-1} \frac{\ln f}{t - \xi}\, dt + \frac{1}{\pi}\int_1^{\infty} \frac{\ln f}{t - \xi}\, dt,$$

$$\Phi(u, \xi) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{t - \xi}\, dt, \qquad |\xi| \geqslant 1; \tag{1.6}$$

we obtain the equation of the free boundary,

$$z = z_0 + \frac{\varphi_0 c^2}{V_0} \int_{-1}^{\xi} \frac{e^{-u}}{(c^2 + \xi^2)^{3/2}}\, e^{+i[p(\xi) + J(u, \xi)]}\, d\xi,$$

$$p(\xi) = \frac{1}{\pi}\int_{-\infty}^{-1} \frac{\ln f}{t - \xi}\, dt + \frac{1}{\pi}\int_1^{\infty} \frac{\ln t}{t - \xi}\, dt,$$

$$J(u, \xi) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{t - \xi}\, dt, \qquad |\xi| \leqslant 1. \tag{1.7}$$

Here, $z_0 = x_0 + iy_0$ is the coordinate of separation point A. The drag of the arc

$$X_0 = \int_0^{S_0} (p - p_0) \sin\theta\, dS. \tag{1.8}$$

Here, $p(S)$ is the pressure distribution on the arc; p_0 is the pressure in the cavity; $\theta(S)$ is the angle between the tangent to the contour at the point having the arc abscissa S and the x-axis. From the Bernoulli integral

$$p - p_0 = {}^1/_2\, \rho\, (V_0^2 - V^2) + \gamma\,(y_0 - y).$$

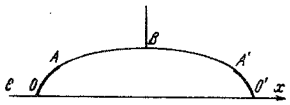

Fig. 1

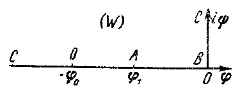

Fig. 3

Substituting into (1.8) the expression for $p - p_0$ and going over to the variable ξ, we obtain

$$X = \frac{1}{2}\rho\varphi_0 V_0 c^2 \times$$

$$\times \int_{-\infty}^{-1} \frac{1 - f^2(s(\xi))}{f(s(\xi))(c^2+\xi^2)^{3/2}} \sin[q(\xi) + \Phi(u, \xi)]\, d\xi + \gamma y_0^2 .$$

Hence

$$c_x = \frac{x}{{}^1\!/_2 \rho V_\infty^2 y_0} =$$

$$= \frac{c^2}{N}\int_{-\infty}^{-1} \frac{1 - f^2(s(\xi))}{f(s(\xi))(c^2+\xi^2)^{3/2}} \sin[q(\xi) + \Phi(u, \xi)]\, d\xi + 4\lambda N,$$

$$N = c^2 \int_{-\infty}^{-1} \frac{\sin[q(\xi) + \Phi(u, \xi)]}{f(s(\xi))(c^2+\xi^2)^{3/2}}\, d\xi . \tag{1.9}$$

To find $u(\xi)$ we construct a nonlinear integrodifferential equation. Writing out the limiting values of the function $F(\zeta)$ on the interval $[-1, 1]$ of the real axis of the ζ-plane in accordance with Sokhotskii's formulas, we obtain

$$v = p(\xi) + J(u, \xi), \qquad |\xi| \leqslant 1 .$$

On a free streamline, where the pressure is constant, the Bernoulli integral has the form:

$$V^2 + 2gy = \text{const} .$$

Differentiating and using the relations

$$V = V_0 e^u, \qquad dy = dS \sin\theta, \qquad dS = \frac{d\varphi}{V}, \qquad \theta = -v,$$

we obtain the equation

$$u(\xi) = \lambda \int_{-1}^{\xi} \frac{c^2}{(c^2+\xi^2)^{3/2}} e^{-3u} \sin[p(\xi) + J(u, \xi)]\, d\xi . \tag{1.10}$$

Here

$$\lambda = \frac{g\varphi_0}{2V_0^3}, \qquad \xi \in [-1, 1].$$

We find the solution of Eq. (1.10) in the space of the continuously differentiable functions C_1. G. N. Pykhteev reduces an extensive group of jet problems to the solution of an equation of the type (1.10), which he calls the general equation of jet theory [5.6]. He has proved a theorem for the existence of a unique solution for the basic equation in jet theory. In relation to our equation this theorem has the form that follows.

Theorem 1. If $\lambda < \lambda_* = c/18.94$, in the sphere Ω_{r_*} there exists the unique solution $u(\xi)$ satisfying the condition $u(-1) = 0$. This solution can be obtained as the limit of the sequence

$$u_0 = 0, \qquad u_{n+1} = \lambda \int_{-1}^{\xi} \frac{c^2}{(c^2+\xi^2)^{3/2}} e^{-3u} \sin[p(\xi) + J(u_n, \xi)]\, d\xi ,$$

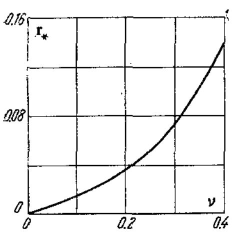

Fig. 4

whose rapidity of convergence is estimated from the inequality $\| u - u_n \|_{C_1} \leqslant (\lambda / \lambda^*)^n r_*$. The quantity r_* is found from the graph shown in Fig. 4, where $\nu = 7.519\, \lambda/c$.

For any fixed, sufficiently small value of the parameter λ Theorem 1 guarantees the existence and uniqueness of a solution for basic equation (1.10) and gives a method of solution. However, the region of values for λ is small. For a somewhat broader region of values for λ it is possible to prove a nonconstructive existence theorem.

Theorem 2. For any $\lambda < c/16.31$, integral equation (1.10) has at least one continuously differentiable solution $u(\xi)$ satisfying the condition $u(-1) = 0$.

Proof. Let Ω_r be a sphere in the space C_1 of continuously differentiable functions defined on the interval $[-1, 1]$ and satisfying the conditions $\| u(\xi) \|_{C_1} \leqslant r$, and $u(-1) = 0$. Here

$$\| u(\xi) \|_{C_1} = \max_{[-1,1]} | u'(\xi) | .$$

Then, obviously, $|u(\xi)| \leq 2r$ on $(-1, 1)$. The sphere Ω_r is a closed set in Banach space. The operator

$$Tu = \lambda \int_{-1}^{\xi} \frac{c^2}{(c^2+\xi^2)^{3/2}} e^{-3u} \sin[p(\xi) +$$

$$+ J(u, \xi)]\, d\xi, \qquad \xi \in [-1, 1]$$

operates from Ω_r in the space C_1. It is easy to see that

$$\| Tu(\xi) \|_{C_1} \leqslant \lambda (1/c) e^{6r} .$$

If $\lambda < rc / e^{6r}$, Tu maps Ω_r onto itself. The ratio r / e^{6r} has a maximum at $r = 1/6$, which is equal to 1/16.31.

We will show that the operator Tu is perfectly continuous. By definition

$$\| Tu_1 - Tu_2 \|_{C_1} = \max \Bigg| \frac{c^2}{(c_2+\xi^2)^{3/2}} \times$$

$$\times \{e^{-3u} \sin[p(\xi) + Ju_1] - e^{-3u_2} \sin[p(\xi) + Ju_2]\} \Bigg| .$$

Inside the braces on the right-hand side we add and subtract the expression

$$e^{-3u_1} \sin[p(\xi) + Ju_2] .$$

We can write the inequalities

$$| e^{-3u_1} - e^{-3u_2} | \leqslant e^{6r}\, 3\, | u_1 - u_2 | \leqslant 6 e^{6r} \| u_1 - u_2 \|_{C_1} ,$$

$$| \sin[p(\xi) + Ju_1] - \sin[p(\xi) + Ju_2] | \leqslant \beta \| u_1 - u_2 \|_{C_1} .$$

The quantity $\beta = {}^1\!/_2 \pi (1 + 2 \ln 2)$ was found by G. M. Pykhteev [6]. Then

$$\| Tu_1 - Tu_2 \|_{C_1} \leqslant (6 + \beta) e^{6r} \frac{\lambda}{c} \| u_1 - u_2 \|_{C_1} ,$$

and this proves continuity of the operator Tu.

The mapping Tu transforms each bounded set of the space C_1 into a bounded set of the space C_1^α for functions whose first derivative satisfies a Hölder condition with the same exponent α as that with which the function $f(s)$ satisfies a Hölder condition. This mapping is compact.

Consequently, the operator Tu is perfectly continuous and maps a convex set of Banach space onto itself. Then confirmation of the theorem follows from the Schauder principle.

We now determine the constants entering into the solution of the problem. From the contour in Eq. (1.6) it follows that the arc length

$$S_0 = \frac{\varphi_0}{V_0} N_1, \qquad N_1 = c^2 \int_{-\infty}^{-1} \frac{dt}{f(s(t))(c^2+t^2)^{3/2}} . \tag{1.11}$$

Then the Froude number, calculated from S_0, is

$$\mathrm{Fr} = \frac{V_\infty}{V_0} \frac{1}{\sqrt{2\lambda N_1}} . \tag{1.12}$$

Satisfying the Bernoulli integral at two points of the zero streamline—at infinity and at the point of convergence of the jet—and using the expression for the Froude number Fr and the cavitation number we have:

$$\sigma = \frac{p_\infty - p_0}{{}^1\!/_2 \rho V_\infty^2} = \frac{V_0^2}{V_\infty^2} + \frac{2}{\mathrm{Fr}^2} \frac{y_0}{S_0} - 1 . \tag{1.13}$$

Here, y_0 is the ordinate of separation point A and is found from (1.6).

Lastly, from the condition at infinity in the physical plane, using (1.4) we find

$$\ln\frac{V_\infty}{V_0}=\frac{c}{\pi}\int_{-\infty}^{-1}\frac{\ln f}{t^2+c^2}dt+ \\ +\frac{c}{\pi}\int_{-1}^{1}\frac{u(t)}{t^2+c^2}dt+\frac{c}{\pi}\int_{1}^{\infty}\frac{\ln f}{t^2+c^2}dt. \quad (1.14)$$

Thus, the seven parameters λ, F, σ, c, V_∞/V_0, y_0, and φ_0/V_0 must satisfy conditions (1.11), (1.12), (1.13), and (1.14). We assign the physical parameters V_∞/V_0, Fr, and S_0. It is easy to see that the above system of equations for the parameters λ, c, φ_0/V_0 and σ will be solvable if it is possible to solve Eq. (1.14) for c. We rewrite Eq. (1.14) in the form:

$$c=R(c)\qquad R(c)=-\frac{\pi}{J}\left(\ln\frac{V_0}{V_\infty}+\frac{c}{\pi}\int_{-1}^{1}\frac{u(t)}{t^2+c^2}dt\right), \\ J=\int_{-1}^{1}\frac{\ln f(s(t))}{1+c^2t^2}dt. \quad (1.15)$$

Theorem 3. Let the conditions of Theorem 1 be satisfied. Then for any δ and α such that $0<\delta<\alpha<\infty$ and for the ratios V_0/V_∞ satisfying the condition

$$+\frac{\delta}{\pi}J+\frac{2r}{\pi}\operatorname{arc\,tg}\frac{1}{\delta}\leqslant\ln\frac{V_0}{V_\infty}\leqslant\frac{\alpha}{\pi}J-\frac{2r}{\pi}\operatorname{arc\,tg}\frac{1}{\delta}, \quad (1.16)$$

Eq. (1.15) has at least one solution $C\in[\delta,\alpha]$.

We first show that the solution of Eq. (1.10), which $\lambda(c)$ is given by the formula

$$\lambda=\frac{V_\infty^2}{V_0^2}\frac{1}{2F^2}\frac{1}{N_1(c)},$$

depends continuously on the parameter c.

Let the value of c_1 correspond to the function $u_1(\xi)$ and the value of c_2 to the function $u_2(\xi)$. As a result of rather clumsy calculations we obtain the inequality

$$|u_1-u_2|<\frac{Me^{6r}}{1-0.461e^{6r}}|c_1-c_2|$$

where M is a positive constant. Hence there follows continuity of the function $u(\xi,c)$ with respect to the parameter c, since $r<r_*=0.139$ on the basis of Theorem 1.

The operator R(c) maps the sphere $K=\{c;\ \|c\|\leq(\alpha-\delta)/2\}$ of the space R_1 into a closed set of elements of the same space bounded with respect to the norm by the quantity

$$\frac{\pi}{J}\left(\lg\frac{V_0}{V_\infty}+2r\operatorname{arc\,tg}\frac{1}{\delta}\right)+\frac{\delta+\alpha}{2}.$$

Here $\|c\|=|c-{}^1\!/_2(\delta+\alpha)|$. Moreover, in view of inequalities (1.16) the operator R(c) possesses the property $\|R(c)\|\leqslant{}^1\!/_2(\alpha-\delta)$, if $\|c\|={}^1\!/_2(\alpha-\delta)$.

The space R_1 is Euclidean. Any bounded set of elements of Euclidean space is compact. Consequently, the operator R(c) is continuous and compact, i.e., perfectly continuous. Then, on the basis of the generalized Schauder principle [12], a fixed point exists in the sphere, and this proves the theorem.

Let, for example, $f(s)$ be such that $f(s(t^{-1}))=[f_1(t)]^{1+c^2t^2}$, where $f_1(t)$ is a positive, continuous, monotonically increasing function satisfying a Hölder condition and the conditions $f_1(0)=0$ and $f_1(1)=1$. Then the upper boundary for the region of variation for $\ln(V_0/V_\infty)$ is strictly greater than the lower (inequality (1.16)), if for any $\delta>10$ we select

$$\alpha>\delta+4r\operatorname{arc\,tg}\frac{1}{\delta}\left(\int_{-1}^{1}|\lg f_1(t)|dt\right)^{-1}.$$

§2. Particular case. We specify the dependence of the velocity at the arc on the arc coordinate in parametric form $V=V_0F_1(n)$ and $S=S_0F_2(n)$, where $F_1(n)$ and $F_2(n)$ are single-valued positive functions of the variable $n\in[n_1,n_2]$, satisfying a Hölder condition and the conditions

$$F_1(n_1)=F_2(n_1)=0,\qquad F_1(n_2)=F_2(n_2)=1.$$

Then the function $f(s(\xi))$, in whose terms the general solution was constructed,

$$f(s(\xi))=F_1(n(\xi)),$$

and $n(\xi)$ is found from the condition

$$\frac{\varphi_0\xi}{\sqrt{c^2+\xi^2}}=-\varphi_0+V_0S_0\int_{n_1}^{n}F_1(n)F_2'(n)\,dn,\quad -\infty<\xi<-1.$$

Let, for example,

$$F_1=\sqrt{\frac{1-n}{1+n}}\qquad F_2(n)=-\frac{1}{A}\int_1^n\frac{(n+1)n\,dn}{(n^2+\beta^2)^{3/2}\sqrt{1-n^2}},$$

$$A=\int_0^1\frac{(n+1)n\,dn}{(n^2+\beta^2)^{3/2}\sqrt{1-n^2}},\qquad \beta=\frac{\sqrt{1+c^2}}{c}.$$

Then

$$n=\frac{\sqrt{\xi^2-1}}{\xi},\qquad f(s(\xi))=\left(\frac{\xi-\sqrt{\xi^2-1}}{\xi+\sqrt{\xi^2-1}}\right)^{1/2}.$$

Substituting $f(s(\xi))$ into the formulas for $q(\xi)$ and $p(\xi)$, we obtain

$$q(\xi)=-\frac{\pi}{2}(\operatorname{sign}\xi),\qquad |\xi|\geqslant 1$$

$$p(\xi)=-2\operatorname{arc\,tg}\left(\frac{1}{\xi}-\frac{\sqrt{1-\xi^2}}{\xi}\right),\qquad |\xi|\leqslant 1.$$

We construct the solution of basic equation (1.10) in the first approximation from the solution of the problem for an imponderable fluid $u_0(\xi)=0$:

$$u_1(\xi)=-\lambda c^2\left[\frac{1}{(c^2+\xi^2)^{3/2}}-\frac{1}{(1+c^2)^{1/2}}\right].$$

We write from (1.6) the equation of the contour,

$$x=-\frac{\varphi_0c^2}{V_0}\int_{-\infty}^{\xi}\frac{\xi+\sqrt{\xi^2-1}}{(c^2+\xi^2)^{3/2}}\sin\Phi(\xi)\,d\xi,$$

$$y=-\frac{\varphi_0c^2}{V_0}\int_{-\infty}^{\xi}\frac{\xi+\sqrt{\xi^2-1}}{(c^2+\xi^2)^{3/2}}\cos\Phi(\xi)\,d\xi,$$

and from (1.7) the equation of the free surface,

$$x=x_0+\frac{\varphi_0c^2}{V_0}\int_{-1}^{\xi}\frac{e^{-u}}{(c^2+\xi^2)^{3/2}} \\ \times\cos\left[-2\operatorname{arc\,tg}\frac{1-\sqrt{1-\xi^2}}{\xi}+\Phi(\xi)\right]d\xi,$$

$$y=y_0+\frac{\varphi_0c^2}{V_0}\int_{-1}^{\xi}\frac{e^{-u}}{(c^2+\xi^2)^{3/2}} \\ \times\sin\left[-2\operatorname{arc\,tg}\frac{1-\sqrt{1-\xi^2}}{\xi}+\Phi(\xi)\right]d\xi.$$

Here,

$$\Phi(\xi)=-\frac{\lambda c^2}{\pi}\left[\frac{1}{(c^2+\xi^2)^{1/2}}\right. \\ \times\ln\left|\frac{(1-\xi)[c^2-\xi+\sqrt{(1+c^2)(c^2+\xi^2)}]}{(1+\xi)[c^2+\xi+\sqrt{(1+c^2)(c^2+\xi^2)}]}\right|+ \\ \left.+\frac{1}{\sqrt{1+c^2}}\ln\frac{1+\xi}{1-\xi}\right].$$

From the equation of the contour it is clear that in an imponderable fluid, when $\Phi(\xi)\equiv0$, the contour is a plate arranged at right angles to the flow.

The system of equations for the parameters takes the form:

$$S_0=\frac{\varphi_0}{V_0}N_1,\qquad N_1=\frac{c^2}{\sqrt{1+c^2}}+ \\ +\sqrt{1+c^2}\,E\left(\frac{c}{\sqrt{1+c^2}}\right)-\frac{1}{\sqrt{1+c^2}}K\left(\frac{c}{\sqrt{1+c^2}}\right),$$

$$\sigma=\frac{V_0^2}{V_\infty^2}+\frac{2}{\mathrm{Fr}^2}\frac{y_0}{N_1}-1,\qquad \lambda=\frac{V_\infty^2}{V_0^2}\frac{1}{N_1\mathrm{Fr}^2},$$

$$\ln\frac{V_\infty}{V_0}=\frac{1}{2}\ln\frac{\sqrt{1+c^2}-c}{\sqrt{1+c^2}+c}-$$

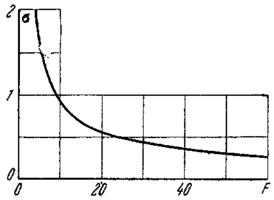

Fig. 5

$$-\frac{2\lambda c}{\pi\sqrt{1+c^2}}\left(1-c\,\mathrm{arc\,tg}\,\frac{1}{c}\right). \tag{2.1}$$

If we set λ = const in (2.1) and let c take values from zero to infinity, then from (2.1) we obtain the parametric function $Fr(\sigma)$. Thus, the graph of function $Fr(\sigma)$ is given in Fig. 5 for $\lambda = {}^1/_{2},c$. The curves corresponding to smaller values of λ lie above the given curve. Consequently, points lying above the curve shown in Fig. 4 determine the region of values for the cavitation numbers and Froude numbers for which a unique solution of Eq. (1.10) exists.

Let δ = 0.1 and α = 1. Then, if $S_0 > 0$, $(V_0/V_\infty) \in [1.28;\ 2.09]$ and $\mathrm{Fr} > 74.2$, on the basis of Theorem 3 Eq. (2.1) is solvable for φ_0/V_0, λ, σ, and c, where $c \in [0.1;\ 1]$.

§3. Cavitational flow in the lower half-plane. If we reflect the flow region shown in Fig. 1 onto the lower half-plane and assume, as before, that the gravitational acceleration vector is directed vertically downward, we obtain a flow pattern analogous to the starting pattern, but for the lower hald-plane. However, the same object is achieved more simply by changing to the reciprocal vector. Then the general solution of the problem for the lower half-plane takes exactly the same form as for the upper half-plane, the only difference being that now the parameter λ has a minus sign, because the Bernoulli integral at the free surface has the form:

$$V^2 - 2gy = \mathrm{const}\,.$$

In particular, the theorem of existence for a solution of basic equation (1.10) remains valid.

Thus, solutions of Eq. (1.10) at λ < 0 correspond to flows in the lower half-plane and solutions at λ > 0 to flows in the upper half-plane.

§4. Limiting case. Let the arc contract into a point. We see that there are nontrivial solutions to the problem of the flow for a ponderous fluid in the upper half-plane part of whose boundary the pressure remains constant, while, as before, the rest of the boundary is straight and horizontal (Fig. 6). This case cannot be obtained as a particular case from the general solution of the problem.

We take as the parametric region the upper half-plane $\zeta = \xi + i\eta$ (Fig. 7). Assuming that the streamline COBO'C is the zero streamline and that the velocity potential at the point O is equal to zero and at the point O' equal to φ_0, we easily see that

$$W = (\varphi_0/2)\ (\zeta + 1)\,. \tag{4.1}$$

The Joukowski function

$$F = \lg\left(\frac{1}{V_0}\frac{dW}{dz}\right),$$

where V_0 is the velocity at the point O, is analytic in the upper half-plane ζ and satisfies the conditions that follow on the real axis ξ:

$$\mathrm{Im}\,F(\xi) = 0,\quad |\xi| \geqslant 1,\quad \mathrm{Re}\,F(\xi) = u = \lg(V/V_0),\ |\xi| \leqslant 1\,.$$

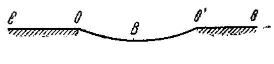

Fig. 6

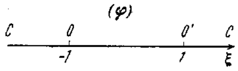

Fig. 7

The function $F(\zeta)$ is found from the solution of the mixed boundary-value problem of the theory of analytic functions. The solution of the problem is bounded at both ends:

$$F(\zeta) = \frac{\sqrt{\zeta^2-1}}{\pi}\int_{-1}^{1}\frac{u(t)}{t-\zeta}\frac{dt}{\sqrt{1-t^2}}\,. \tag{4.2}$$

Repeating all procedures that led to Eq. (1.10), we obtain

$$u'(\xi) = \lambda e^{-3u}\sin Iu\,,$$
$$Iu = \frac{\sqrt{1-\xi^2}}{\pi}\int_{-1}^{1}\frac{u(t)}{t-\xi}\frac{dt}{\sqrt{1-t^2}}\,,\qquad |\xi| \leqslant 1. \tag{4.3}$$

Solving Eq. (4.3) for $u(\xi)$, from (4.1) and (4.2) we obtain the general solution of the problem.

Equation (4.3) is nonlinear and homogeneous, i.e., u = 0 is its solution. It proves to have bifurcation points. We find the Frechet derivative of the nonlinear operator in the right-hand side of Eq. (4.3) and construct the equation

$$u'(\xi) = \lambda Iu,\quad |\xi| \leqslant 1\,. \tag{4.4}$$

In linear equation (4.4) we make the change of variables ξ = cos σ and then integrate the right-hand side by parts. The integrated part is equal to zero and Eq. (4.4) takes the form:

$$u'(\sigma) = \lambda\,\frac{\sin\sigma}{\pi}\int_0^\pi u'(\delta)\ln\left|\frac{\sin{}^1/_2(\delta+\sigma)}{\sin{}^1/_2(\delta-\sigma)}\right|d\delta,\qquad \sigma\in[0,\pi]\,. \tag{4.5}$$

The Fredholm operator with kernel

$$K(\sigma,\delta) = \ln\left|\frac{\sin{}^1/_2(\sigma+\delta)}{\sin{}^1/_2(\sigma-\delta)}\right|$$

will be perfectly continuous [9]. The second integrated kernel

$$K_2(\delta,\sigma) = \int_0^\pi \sin\delta\,\ln\left|\frac{\sin{}^1/_2(\delta+t)}{\sin{}^1/_2(\delta-t)}\right|\sin t\,\ln\left|\frac{\sin{}^1/_2(t+\sigma)}{\sin{}^1/_2(t-\sigma)}\right|dt$$

possesses the property that, on the entire interval (0, π), $K_2(\delta,\delta)$ is greater than zero and vanishes only at the ends of the interval. Then from a familiar theorem [10] Eq. (4.5) has a unique nonnegative fundamental function. The corresponding eigenvalue is positive, simple, and less than the modulus of any other eigenvalue of this equation. When the eigenvalues of the linear equation are odd-multiple, it is legitimate to linearize the nonlinear equation to determine its bifurcation points [11]. Consequently, the starting equation (4.3) has at least one bifurcation point. All other bifurcation points of Eq. (4.3), if there are any, will correspond to flow in the lower half-plane.

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22 June 1967 Novosibirsk